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# OPTIMUM DESIGN OF INFINITE JOURNAL BEARING WITH A MINIMUM OF THE FRICTION MOMENT†

V. I. GRABOVSKII and A. N. KRAIKO

Moscow

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The isoperimetric problem (IP) of profiling the optimum outer shape of the gap of a closed hydrodynamic journal bearing of infinite length is formulated and solved in the incompressible "non-cavitating" fluid approximation. Whereas the maximum of the carrying capacity coefficient $C_N$ is realized in the Rayleigh problem (RP), in the IP a minimum of the coefficient of the friction moment $C_M$ on the neck (shaft) for given $C_N$ is achieved. The structure of the optimum solution is established and it is shown that if $C_N$ is less than the coefficient $C_{NR}$ corresponding to the RP, the optimum gap height $h$ is a continuous function of the polar angle $\theta$. In the general case the optimum function $h = h(\theta)$ contains segments of four kinds. Two of them, $h \equiv 1$ and $h \equiv H > 1$, are boundary extremum segments (BES*1*) and BES*H*), which appear due to the fact that $h$ has lower and upper bounds ($h$ is measured relative to the minimum admissible height). The other two segments are two-sided extremum segments—TES. The first of these, TES*1*, is similar to the TES in Rayleigh's problem, in which $h \equiv h_1$ where $1 < h_1 < H$. TES2 appears only in the IP. For $C_N > 0.75C_{NK}$ BES*1* divides TES2 in two. The first part has a negative slope, and the second has a positive slope and connects BES*1* with BES*H* or TES*1*. As $C_N \to C_{NR}$ the slopes of the tangents to TES2 approach $\pm\infty$, and the segments themselves turn into two steps, i.e. into the well-known discontinuities of $h$ in the RP. Unlike the IP for a slider, the optimum gap of a journal bearing on TES2 can either be converging or diverging. Results of calculations are given to illustrate the theoretical analysis.


Improving hydrodynamic journal bearings, widely used in various applications [1–3], involves the problem of profiling the optimum gap. Rayleigh was the first to carry out research on this problem [4]: in the approximation of an incompressible viscous fluid he found that the maximum of $C_N$ of a cylindrical slider bearing is given by a piecewise-constant gap with one step. Over the initial segment TES*1* the gap height $h \equiv h_R > 1$ satisfies Euler's equation. The terminal segment $h \equiv 1$, where $h$ is divided by the minimum admissible height $h_m$ according to the formulation of the problem, is a boundary extremum segment (BES*1*). The solution of the RP for a journal bearing also has a step structure [5–9].

Other interesting variational problems, apart from the RP, include the minimization of the drag coefficient $C_D$ or friction moment $C_M$ for fixed $C_N$. The first attempt at its solution for a slider was made in [10], and completed in [11]. A similar isoperimetric problem for an infinite journal bearing is formulated and solved below.

**1.** Let $r$, $\theta$, $z$ be cylindrical coordinates with $z$ axis in the direction of the axis of the bearing shaft. The shaft of radius $R$ rotates counter-clockwise with angular velocity $\omega$. The equation of the generator of the fixed cylindrical "base" of the bearing (Fig. 1a) is $r = R_1(\theta)$. The gap height $h(\theta) = R_1(\theta) - R$ is such that $0 < h \leqslant R$. In variables $x = \theta/(2\pi)$ and $y = r - R$ $(0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant h)$, the equations of flow in the gap are the same as those in the gap of a two-dimensional slider moving over an infinite plane. The difference lies in the expressions for the forces and in some of the boundary conditions. The gap shape is defined by its "upper" boundary $h(x)$, which is the bearing base. In general, $h(x)$ can have a step (or steps) at $x = x_d$ (Fig. 1b). We shall denote quantities at points $0, f, d, \ldots$ by appropriate subscripts. If the variables have a discontinuity at $d$, we shall use an additional subscript minus (plus) before (after) it in the direction of rotation. The viscosity of the fluid $\mu$ and its density $\rho$ are constant.

We introduce dimensionless variables, taking as the scale of $r$, $y$ and $h$, the peripheral component of the velocity $u$, the density and the pressure, the dimensional radius of the shaft $R$, the minimum admissible gap height $h_m$, $U = R\omega$, $\rho$ and $\gamma\rho U^2$ with the dimensionless complex

$$\gamma = 12\pi R\mu/(\rho h_m^2 U) = 12\pi\mu/(\rho h_m^2\omega) = 6\mu/(\rho h_m^2 n)$$

where all the variables are dimensional, $\omega = 2\pi n$ and $n$ is the number of rotations in unit time.

In lubrication theory with piecewise-continuous function $h(x)$, the pressure is a continuous function of $x$ which is independent of $y$. We shall use the equation for $p$ in two forms

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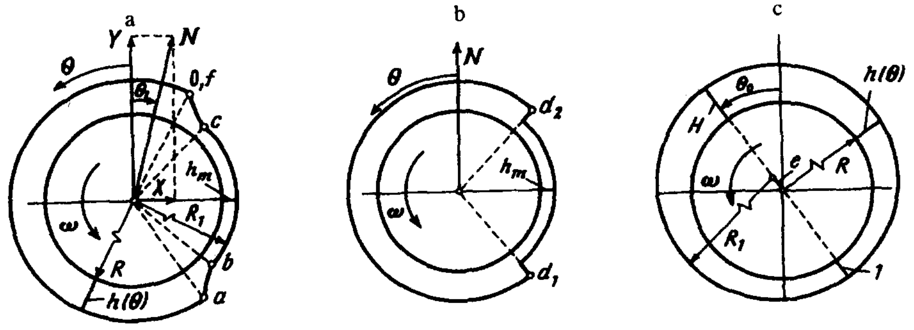


Fig. 1.

$$1)\ L \equiv h - q - p'h^3 = 0, \quad 2)\ p' = \frac{h-q}{h^3}; \quad q = 2Q \equiv 2\int_0^h u dy \tag{1.1}$$

Here $Q$ is the lubricant flow rate through the clearance, and the prime denotes differentiation with respect to $x$. For any given piecewise-continuous function $h(x)$ the constant $p$ is chosen to satisfy the condition for periodic pressure

$$p(0) = p(1) \equiv p_0 \tag{1.2}$$

In the RP [5–9] the pressure $p_0$ or, to be more precise, the pressure $p_d$ in one of the sections of the gap jump is assumed given. However, as we shall see, this is not necessary.

Let $Y$ and $X$ be the dimensional vertical and lateral components of the force **N** acting on the shaft due to the non-uniform pressure distribution $p$ over its surface, and let $M$ be the friction moment (Fig. 1a). The coefficients $C_Y$, $C_X$ and $C_M$ are given by

$$C_Y \equiv \frac{Y}{2\pi R\gamma\rho U^2} = -\int_0^1 p\cos\theta dx, \quad C_X \equiv \frac{X}{2\pi R\gamma\rho U^2} = \int_0^1 p\sin\theta dx$$
$$C_M \equiv \frac{M}{Rh_m\gamma\rho U^2} = \frac{1}{2}\int_0^1 \left(\frac{1}{3h} + hp'\right)dx; \quad \theta = 2\pi x \tag{1.3}$$

The expression for $C_M$ is identical to the expression for the drag $C_D$ of an infinite slider bearing [11]. There are consequently a number of analogies and similarities in the solutions of the two problems.

In the RP we seek a distribution of the gap height $h = h(x)$ which, for $p$ defined by Eq. (1.1) and condition (1.2), gives a maximum of the carrying capacity coefficient $C_N = \sqrt{(C_Y^2 + C_X^2)}$. In the IP $C_N$ is fixed and $C_M$ is minimized. Of course, $C_N \leqslant C_{NR}$, where $C_{NR}$, is the largest possible carrying capacity coefficient obtained in the RP.

In both problems the clearance height consistent with the choice of scale has a lower and upper limit

$$1 \leqslant h(x) \leqslant H \tag{1.4}$$

with a given constant $H > 1$.

We should emphasize that the dimensional minimum admissible gap height $h_m$ stipulated in the formulation of the problem is determined by considerations of a physical nature (the surface roughness), the presence of solid impurities in the lubricant, possible vibration of the bearing shaft, etc.). According to [11], in the IP for small $C_N$, the minimum height of the optimum gap of a two-dimensional bearing $h_{\min} > h_m$. It will become clear below that the same applies in the IP for the optimum gap of a journal bearing. On the other hand; in the RP and for large $C_N$ in the IP $h_{\min} > h_m$. This is easily explained. In fact, in the RP in the approximation of lubrication theory if there is no specified $h_m$ for which a decrease is inadmissible owing to "external" constraints, the carrying capacity $N$ of a slide or bearing increases without limit as $h_{\min} \to 0$ without optimum profiling of the gap, so that the variational problem actually becomes meaningless. Of course, the fact that $N$ increases without limit as $h_{\min} \to 0$ is the result of ignoring surface roughness and other effects, and the existence of these makes it necessary to specify $h_m$. Incidentally, it is only if $h_m$ is taken as the scale of gap height that there can be a BES*1* on which $h = 1$, and we can have $\delta h \geqslant 0$. But if, in the absence of $h_m$, $h$ is taken relative to $h_{\min}$, then, for $h = 1$, admissible values of $\delta h$ can be of any sign, and there can be no BES*1*. Although this was appreciated by Rayleigh, it is unfortunately not always understood by present-day authors. The consequence of this mistake [10] was analysed in [11]. It applies for similar reasons to the variable $H$, which is also defined from "external" considerations.

**2.** To solve the IP and the RP we write the Lagrange functional

$$J=\alpha C_M+\beta C_N+\int_0^1 \lambda(x)L dx;\quad C_N=\sqrt{C_Y^2+C_X^2},\quad \frac{C_Y}{C_N}=\cos\theta_1 \tag{2.1}$$

where $L$ is the left-hand side of the first equation of (1.1), $\lambda$ is a variable Lagrange multiplier, $\alpha = 0$ and $\beta = 1$ in the RP and $\alpha = 1$ and $\beta$ is a constant Lagrange multiplier in the IP. The coefficient $C_N$ corresponds to the resultant force acting on the shaft in the direction $\theta = \theta_1$. In both problems for admissible variation the variations of $J$ and the optimized functional of (1.3) are the same for any limited Lagrange multipliers. As a result, for the optimum gap in the RP $\delta J = \delta C_N \leqslant 0$ and in the IP RP $\delta J = \delta C_M \geqslant 0$ for any variation of $h$ which satisfies conditions (1.4). By varying $J$ and taking account of the continuity of $p$ in sections $d$ of a possible jump of $h$ and the known rules for determining the variations, we arrive at an expression for $\delta J$ which holds for any (not necessarily optimum) gap height $h(x)$ and arbitrary limited $\lambda(x)$ and $\beta$

$$\delta J=\{\alpha(h_- - h_+)/2+(\lambda h^3)_+ - (\lambda h^3)_-\}_d\Delta p_d+X_d\Delta x_d-\Delta q\int_0^1 \lambda dx+\int_0^1 (A^h\delta h+A^p\delta p)dx \tag{2.2}$$

$$X_d=\alpha(h_-^{-1}-h_+^{-1})/6+\lambda_-(h-q)_- - \lambda_+(h-q)_+$$

$$A^h=AB/(6h^3),\quad A=2h-3q,\quad B=\alpha-6\lambda h^2$$

$$A^p=-\beta\cos(\theta-\theta_1)-(\alpha h-2\lambda h^3)'/2$$

Here $\Delta p_d$ and $\Delta x_d$ are the differences between the values of $p$ and $x$ over the sections corresponding to the jump of $h$ for the varied and non-varied gaps, while $\delta p$ and $\delta h$ are the variations of $p$ and $h$, that is, the differences for the same gaps for fixed $x$, the terms with $\Delta p_d$ and $\Delta x_d$ being summed over all sections of the jump of $h$. The coefficients $X_d$, $A$ and $B$ are transformed taking account of the expressions for $p'_\pm$ from (1.1).

Using the arbitrariness in the choice of $\lambda$, we make the coefficient $A^p$ equal to zero. This leads to the first-order ordinary differential equation

$$(\alpha h-2\lambda h^3)'=-2\beta\cos(\theta-\theta_1) \tag{2.3}$$

which holds for any gap over the continuity segments of $h$. If there are $k$ sections of discontinuity of $h$, we can obtain conditions relating $\lambda_{d-}$ to $\lambda_{d+}$ in all, apart from one of these, by equating the coefficient of $\Delta p_d$ to zero. This gives

$$\alpha(h_- - h_+)_d+2[(\lambda h^3)_+ - (\lambda h^3)_-]_d=0 \tag{2.4}$$

By (2.4) the expression in brackets in (2.3) is continuous in the ($k$-1)th of $k$ sections of the discontinuity of $h$. When $x$ changes by one, $\theta$ changes by $2\pi$. Thus, integrating (2.3) from the point $d_+$ to the point $d_-$, corresponding to the $k$th jump $h$, and allowing for condition (2.4) in other sections of discontinuity of $h$, we find that the same condition is satisfied on the $k$th jump $h$. By condition (2.4), expression (2.2) for $\delta J$ does not contain any of the increments $dp_d$ at any point of a jump of $h$. It is therefore unnecessary to fix $p_d$ of $p_0$ when formulating the RP and IP.

Conditions (2.4) at points of discontinuity of $h$ are insufficient for determining $\lambda$. The missing condition can be obtained by equating the coefficient of $\Delta q$ to zero

$$\int_0^1 \lambda dx=0 \tag{2.5}$$

Equations and conditions (2.3)–(2.5) state the adjoint problem for the multiplier $\lambda$.

Suppose $h(x)$ is a given function. We can integrate Eq. (1.1) for arbitrary $p(0) = p_0$ and choose the flow rate $q$, solving the direct problem of lubrication theory to satisfy the condition of periodic pressure from (1.2) $p(1) = p_0$ in a cyclic circuit of the shaft. In the case of an incompressible lubricant Eq. (1.1) contains $p'$ but not $p$. The value obtained for $q$ and all the other results are thus independent of the choice of $p_0$. Once we have solved the direct problem, we can solve the adjoint problem where, in a solution of (2.3) which is linearly dependent on $\lambda_0 = \lambda(0)$, $\lambda_0$ can always be chosen to satisfy condition (2.5). Then for any (not necessarily optimum) gap the expression for $\delta J$ will become

$$\delta J = X_d \Delta x_d + \int_0^1 A^h \delta h dx \tag{2.6}$$

$$X_d = \left\{ \frac{\alpha}{6h_\pm}[3qh_\pm + h_\mp(h_\mp - 3h_\mp)] + \lambda_\pm f \right\} \frac{h_+ - h_-}{h_\mp^3}$$

$$f = h_+h_-(h_+ + h_-) - (h_+^2 + h_+h_- + h_-^2)q$$

$$A^h = AB/(6h^3), \quad A = 2h - 3q, \quad B = \alpha - 6\lambda h^2$$

Both of the equivalent representations of the coefficient $X_d$ (with upper or lower sign in subscripts) are obtained from the corresponding coefficient of (2.2) by eliminating either $\lambda_-$ or $\lambda_+$ using condition (2.4), which holds on jumps of $h$.

In the RP in which $\alpha = 0$ and $\beta = 1$, analysis of expression (2.6) shows that in general the optimum gap can consist of segments of three kinds. These are BES*1*, where $h \equiv 1$, BES*H* where $h \equiv H$ and TES*1*. On TES*1*, the value of $h$ is defined by the condition $A^h = 0$, or, what is the same thing, $A = 0$, that is, by the equation

$$2h - 3q = 0 \tag{2.7}$$

By (1.4) admissible $\delta h$ are non-negative on BES*1*, and admissible $\delta h$ are non-positive on BES*N*. Since admissible variation of the optimum gap in the RP can lead only to a reduction of $C_N$, $\delta J = \delta C_N$ are non-positive in that problem, and the optimality conditions of these segments are in the form of inequalities

$$\lambda(3q - 2H) \geqslant 0 \text{ on BES}N$$
$$\lambda(3q - 2) \leqslant 0 \text{ on BES}1 \tag{2.8}$$

In general, different segments can be joined together with or without jump of $h$. For an optimum "discontinuous" joint in the RP the coefficient of $\Delta x_d$ in (2.6) in the section of the jump of $h$ must be zero, i.e. $X_d = 0$. In the RP where $\alpha = 0$ this means that a discontinuous joint between TES*1* and any of the segments of the boundary extremum is possible only if in the section of the jump of $h$

$$\lambda_{d-} = \lambda_{d+} = 0 \tag{2.9}$$

By (2.7) and (1.1), on the TES*1*

$$h = 3q/2, \quad p' = 1/(3h^2) \tag{2.10}$$

Hence for an incompressible lubricant $h$ = const on TES*1*, and $p$ is a linear function of $x$. Using conditions (2.5) and (2.7)–(2.9), it can be shown in the RP that the optimal function $h(x)$ has two discontinuities, both of which lie either between BES*H* and BES*1* if $H < h_R$ and simply no TES*1* exists, or between TES*1* and BES*1* when $H > h_g$ (Fig. 1*b*). Thus, for an incompressible lubricant, in the RP the optimum gap is a step function with two steps. This result is known for $H > h_g$, when the upper constraint on $h$ is unnecessary in the RP [6–9]. At the same time, fixing $p$ in one of the sections of the jump of $h$, which according to the formulation of the problem is superfluous, can lead to ambiguity.

As an example, consider the "second" solution of the RP constructed in [9]. It is actually only nominally different from the "first" solution, the only difference arising from the fact that the scale of $p$ is taken as the pressure at the inlet or outlet (in the direction of the shaft rotation) of BES*1*. In fact, at least for an incompressible lubricant, it is more natural to relate $p$ to $\rho U^2$ rather than to the pressure in any "typical" section, even when such exists (as in the case of a slide or journal bearing with open base).

In the IP $\alpha = 1$ and $A^h$ can vanish not only when $A = 0$ which, as before leads to (2.7) and to TES*1*, but also when $B = 0$. In the second case

$$6\lambda h^2 = 1 \tag{2.11}$$

Thus in the IP part from the TES*1* we can have TES*2* on which $h$, by virtue of (2.3) and (2.11), is given by the equation

$$h' = -3\beta\cos(\theta - \theta_1)$$

Like (2.8), in IP on BES$H$ and BES*1* we must have

$$(1 - 6\lambda H^2)(2H - 3q) \leq 0 \quad \text{on BES}N$$
$$(1 - 6\lambda)(2 - 3q) \geq 0 \quad \text{on BES}1 \tag{2.12}$$

If we substitute $\lambda_+$ or $\lambda_-$ of (2.11) into the formula for $X_d$ of (2.6) with $\alpha = 1$, then, as can be shown, $X_d = 0$ only when $h_{d+} = h_d$. This means that the TES*2* is joined continuously with other segments.

**3.** The solution of the RP gives the largest value of $C_N$ of a bearing and a large (but not the largest) value of $C_M$. When $C_N = 0$, there are two limit shapes of the gap associated with the constraint on its height, $h \equiv H$ and $h \equiv 1$. In the first case the coefficient $C_M$ is a minimum ($C_M = 1/(6H) \to 0$ and $H \to \infty$) and in the second, a maximum ($C_M = 1/6$). As in [11], we start by constructing a solution of the IP in the extreme case where $C_N = 0$. This happens for any $h \equiv$ const, and $C_M$ decreases as the gap height increases. Thus for $C_N = 0$ by virtue of the upper constraint on $h$, the solution of the IP is given by the equation

$$h(x) \equiv H \tag{3.1}$$

It then follows from (1.1) and (1.2) that

$$p(x) \equiv p_0 \tag{3.2}$$

For solution (3.1) to give a minimum of $C_M$, the corresponding condition of (2.12) must be satisfied. Since according to (1.1), (3.1) and (3.2) in this case $h \equiv H - q$, we have $2H - 3q = -q < 0$. Hence the first inequality of (2.12), a necessary condition for an optimal solution (3.1), is satisfied if

$$1 - 6\lambda H^2 \geq 0 \quad \text{for} \quad 0 \leq x \leq 1 \tag{3.3}$$

Having found $\lambda(x)$ from the differentiated equation (2.3) with $h$ from (3.1) and from condition (2.5), we obtain ($\theta_1 = 0$ is achieved by the choice of the origin of $\theta$)

$$\lambda(x) = \beta\sin(2\pi x)/(2\pi H^3)$$

that is, $\lambda$ is a sinusoid of $x$ with variation range $-|\beta| \leq 2\pi\lambda H^3 \leq |\beta|$. Using this fact and (3.3) we can find a range of values of $\beta$ in which the necessary condition for optimality, inequality (3.3), is satisfied. As in [11], the specific value of $\beta$ from this range is chosen by the condition for a continuous transition from $C_N = 0$ to $C_N > 0$. For such a transition to occur, we can have as small a positive coefficient $C_N$ as we please by introducing a small TES*2* into the neighbourhood of $x = x_0 = 0.75$. Hence, for $C_N = 0$ at the same value of $x = 0.75$ inequality (3.3) becomes an equality. Thus $C_N = 0$ corresponds to $|\beta| = \pi H/3$.

Unlike [11], in which a complete analytic solution was given of the IP for an infinite slider, we have solved the IP for a journal bearing numerically. However we have used many of the results obtained in [11]. In the numerical solution of the IP the differential equations are integrated successively over different segments taking into account the continuity of $p$, $h$ and $\lambda$ at the points where they join. It is due to the presence of TES*2*, where $p$ and $h$ can either increase or decrease, that it is possible to construct continuous solutions in the IP.

The parameters of the problem to be determined, according to the equations and boundary conditions for given $H$ and $C_N$, are $q$, $\beta$, $\lambda(0)$ and $C_M$. The solution is independent of the dimensionless complex $\gamma$ because of the scaling method and of the value $p_0$ (for an incompressible lubricant), owing to the presence of only the derivative $p$ in Eq. (1.1) for $p$. Nevertheless a certain value of $p_0$, for example $p_0 = 0$, must be assigned in the calculations. Each of the remaining four parameters must satisfy one of the conditions in the IP. Thus, $\beta$ is chosen to obtain the given $C_N$. Periodic pressure is secured by the choice of $q$. Finally, the value $\lambda_0 = \lambda(0_+)$ is chosen to satisfy condition (2.5). In accordance with the results of Sections 1 and 2 the equations used to construct the above solution have the form

$$h \equiv H, \quad p' = \frac{H-q}{H^3}, \quad \lambda' - \frac{\beta}{H^3}\cos(2\pi x) = 0 \quad \text{on BES}N$$

$$h = \frac{3q}{2}, \quad p' = \frac{1}{3h^2}, \quad \lambda' - \frac{\beta}{h^3}\cos(2\pi x) = 0 \quad \text{on TES}1$$

$$h' = -3\beta\cos(2\pi x), \quad p' = \frac{h-q}{h^3}, \quad \lambda = \frac{1}{6h^2} \quad \text{on TES}2$$

$$h \equiv 1, \quad p' = 1-q, \quad \lambda' - \beta\cos(2\pi x) = 0 \quad \text{on BES}1$$

On each segment of the boundary extremum its optimality condition must be satisfied in the form (2.12). The calculation begins in the "initial" section $x = 0$, passes through all possible segments and ends at $x = 1$. According to the calculations, the section $x = 0$ contains BES*H*, or if it does not exist, TES*1*. When the optimality condition is violated for the current segment, a transition is made to TES2, then to BES*1*, then again to TES2 and, finally, to TES*1* and to BES*H*. This completes the circuit over the shaft perimeter. Certain segments may be absent, depending on the values of $C_N$ and $H$. As already noted, TES2 is only absent when $C_N = C_{NR}$, when it degenerates into a jump of $h$.

The solution of the RP plays a special part in the IP because the value $C_N = C_{NR}$ corresponding to the RP is the maximum value of $C_N = C_{NR}$ assigned in the IP. The solution of the RP, which is especially simple in the case of an incompressible lubricant, was obtained from the solution of the IP by taking the limit as $\beta \to \infty$. This is done as $C_N \to C_{NR}$ by continuously reducing the length and simultaneously increasing the slope of TES2 (in absolute value). Thus, for $\beta = -10^3$, calculation gives

$$q_R \approx 1.2086, \quad h_R \approx 1.813, \quad C_{NR} \approx 0.01345, \quad C_{XR} = 0$$
$$\theta_1 = 0, \quad C_{MR} \approx 0.1441, \quad \theta_{d_1} \approx -0.8276\pi, \quad \theta_{d_2} \approx -0.1724\pi \tag{3.4}$$

These are close to the results obtained in [6–9]. The RP has a meaning for any constraints on $h$, including "rigid" values for which $H < h_g$. In such cases the inlet TES*1* of the optimum gap with $h = h_R$ is replaced by a horizontal BES*H* with $h = H < h_R$.

**4.** Figure 2 gives the parameters of optimum journal bearings designed by the above method. It shows $C_N^0 = C_N/C_{NR}$ and $C_N^0 = C_M/C_{MR}$ with $C_{NR}$ and $C_{MR}$ from (3.4). Since $C_N \leqslant C_{NR}$, $0 \leqslant C_N^0 \leqslant 1$. At the same time $C_M^0$ can be larger than one, because the maximum value of $C_M = 1/6 \approx 0.167$ is obtained for a gap with $h \equiv 1$, corresponding to the IP with $H = 1$. This gives a maximum value of $C_M^0 \approx 1.161$. The numbers next to the curves or points in Fig. 2 indicate the values of $H$. All the curves for $H \geqslant h_R = 1.813$ for certain $C_N^0 \leqslant 1$ reach the "envelope" $E$, which corresponds to the IP when there is no upper limit on $h$ (formally for $H = \infty$, although for $C_N^0 > 0$ the maximum values of $h$ outside a small neighbourhood of the "origin of coordinates" are small). Two such curves are shown for $H = 4$ and $H = 2$. The points where they meet the envelope $E$ are given the numbers 4 and 2. Above these points

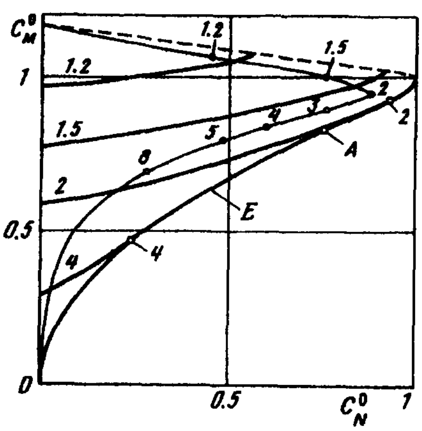


Fig. 2.

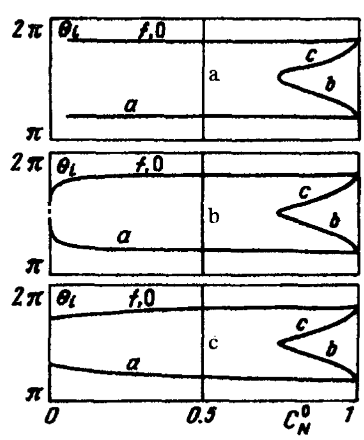


Fig. 3.

the corresponding optimum gaps do not contain BES$H$, that is, the given upper limits on $h$ are unessential. Below the point $A$ on the curve $E$ the optimum gaps do not contain BES$1$. The point $A$ corresponds to $C_N^0 \approx 0.75$. Hence, for $C_N^0 < 0.75$ the lower limit on $h$ is unessential in the solution of the IP. Similarly [11] it can be show that this part of the envelope is given by a "self similar" solution for which $C_N^0 = k(C_M^0)^2$. The calculations give $k = 1.094$.

The value $H < h_R \approx 1.813$ corresponds to another type of solution. The curves do not meet the envelope $E$, but end when $C_M^0 > 1$. There are two of them ($H = 1.5$ and $H = 1.2$) in Fig. 2. Their right-hand points correspond to the RP with the additional condition $h \leqslant H < h_R$. The solution of this RP gives a step gap with $h \equiv 1$ for $x_{d1} < x < x_{d2}$ and with $h \equiv H$ for other $x$, where $x_{d1} \to 0.5$ and $x_{d2} \to 1$ as $H \to 1$. These solutions correspond to the points on the dashed curve.

To compare optimum with non-optimum gaps, we chose the well-known "eccentric" journal bearing [1]. The outer boundary of its gap is formed by a circle with different centre from the shaft (Fig. 1c). Let the shaft radius be $R$, the eccentricity of the bearing (displacement of the centres of the cylinders) be $e \ll R$, and let the axis of displacement be rotated through an angle $\theta_0$. If 1 and $H$ are the minimum and maximum gap heights, then

$$h(\theta) = 1 + e[1 + \cos(\theta - \theta_0)], \quad e = (H-1)/2$$

The results for such bearings are shown by the thin continuous curve in Fig. 2. The numbers refer to the values of $H$. As can be seen, the advantage of optimum gaps increases as $C_N^0$ decreases. The coefficients $C_M^0$ of the optimum and non-optimum bearings were compared for identical $C_N^0$ and $H$. Some of the results are given below

| $H$ | 1.2 | 1.5 | 2 | 3 | 4 | 5 | 8 | 10 | 20 |
|---|---|---|---|---|---|---|---|---|---|
| $C_N^0$ | 0.44 | 0.76 | 0.88 | 0.76 | 0.60 | 0.48 | 0.28 | 0.21 | 0.08 |
| $C_M^0$ | 1.08 | 1.0 | 0.95 | 0.89 | 0.85 | 0.80 | 0.70 | 0.65 | 0.49 |
| $\delta C_M^0$(%) | 3.4 | 4.1 | 5.6 | 8 | 14 | 20 | 38 | 48 | 79 |

Here $\delta C_M^0$ is the (percentage) excess of the coefficient $C_M^0$ of the non-optimum bearing over that of the optimum one. The maximum optimum gap height satisfying the condition $h \leqslant H$ was less than $H$ for $H > 2$. We see that $\delta C_M^0$ increases monotonely as $H$ increases. For values of $H$ for which the carrying capacity of the eccentric bearing is closed to its maximum ($C_N^0 \approx 0.88$), $\delta C_M^0 = 5.6\%$ and increases rapidly as $C_N^0$ decreases, reaching 48% and 79% for $C_N^0 = 0.21$ and 0.08, respectively.

The gap geometry of the optimum bearings is simple, consisting either of two circular segments of constant height connected by two segments of variable height or (for small $C_N$) of one circular segment and one segment of variable height. The coordinates of the boundaries of the segments of different types $\theta_i$ ($x_i = \theta_i/2\pi$ while ($i = 0, a, b, c$ and $f$) are shown in Fig. 3. The notation used in Fig. 3 and below is explained in Fig. 1(a), in which the segment is a TES$1$ (or BES$H$) from 0 to $a$, a narrowing TES$2$ from $a$ to $b$, a BES$1$ from $b$ to $c$, and a widening TES$2$ from $c$ to $f$. According to Fig. 1(a) $\theta_c - \theta_b$ is the angular extent of BES$1$, and $\theta_b - \theta_a$ and $\theta_f - \theta_c$ are the angular extents of the two TES$2$.

Figure 3(a) corresponds to the envelope $E$ of Fig. 2, Fig. 3(b) corresponds to $H = 4$, and Fig. 3c corresponds to $H = 1.2$. There is very little difference between the corresponding curves in the different figures. The greatest difference appears as $C_N^0 \to 0$, when the upper limit on $h$ becomes important. Those of the functions $\theta_i = \theta_i(C_N^0, H)$ that determine the extent of TES$2$ and BES$1$ depend closely on $C_N^0$. The extent and position of TES$1$ are practically constant. The middle of the segment (BES2 or TES$2$) on which the gap height $h$ is a minimum is always at $\theta = 3\pi/2$, and for large $C_N^0$ the coordinates of its end and beginning are close to the values for the RP: $\theta_a \approx 1.17\pi$ and $\theta_f \approx 1.83\pi$. For $0.75 < C_N^0 < 0.75$ the minimum height of the optimum gap is equal to one, and for $0 < C_N^0 < 0.75$ it is greater than one, tending to $H$ as $C_N^0 \to 0$.

For $H = 4$ Fig. 4 shows how the optimum gaps behave in the transition from $C_N = 0$ to $C_N = C_{NR}$. The curves 0, 1, 2, 3, 4, 5, 6 and 7 correspond to $C_N^0 = 0.012$, 0.035, 0.238, 0.421, 0.748, 0.901 and 1. Initially the lower limit has now influence on the gap shape (for $0 < C_N^0 < 0.75$). Curves 3 and 4 correspond to a "self-similar" solution for which neither the lower nor the upper limits on the gap height are unimportant. If $h(x)$ is divided by its minimum or maximum value on them, they coincide. For $0.75 < C_N^0 < 1$ only the lower limit on $h$ affects the shape of the optimum gap (curves 5 and 6). Finally, curve 7 corresponds to the RP. The pressure distributions $p(x)$ for the gaps in Fig. 4 are shown in Fig. 5. The largest pressure is obtained in the RP, which also gives the greatest carrying capacity. For all solutions of the IP the point of minimum pressure always lies on a TES$2$, which is widening in the direction of rotation of the shaft (or part of it), rather than at the origin of the TES$1$, as in the RP.

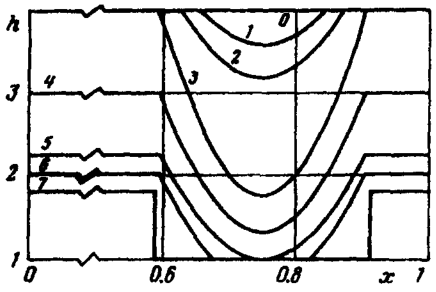


Fig. 4.

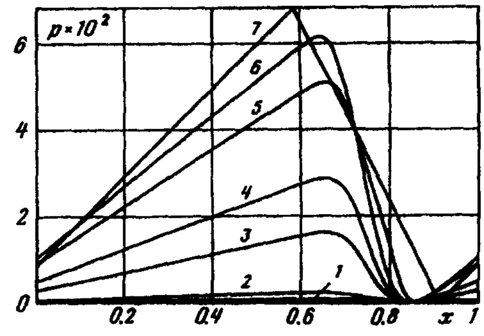


Fig. 5.

Similarly, the point of maximum pressure is always on a narrowing TES2. In Fig. 5 the pressure for each gap is measured from its minimum value.

The geometric and force parameters of gaps that give solutions of the IP for which the upper limit on $h$ is unessential are summarized in the top part of Table 1, in which $h_{\min}$ is the minimum gap height ($h_{\min} \geqslant 1$). Its first row corresponds to the RP. The next five row refer to gaps for which there is an important lower limit on $h$. The other two rows (seventh and eighth without coordinates of the ends of the BES1 $x_b$ and $x_c$) correspond to a self-similar solution for which the lower constraint on $h$ is unimportant. The lower part of the table gives the results for $H = 2$, for which the upper constraint on $h$ is important. They are a continuation of the results in the third row. In the solution corresponding to the ninth row, both the lower and the upper constraints on $h$ are important. Then, up to the limiting case $h \equiv H$ with $C_N = 0$, represented by the last row, the lower constraint on $h$ is unimportant.

We conclude by returning to the question of whether to assign the pressure in a "characteristic" section of a journal bearing. As we have seen, in both the RP and the IP it is superfluous to do so in the case of a journal bearing with a closed base. Although in [6–9] the pressure in one of the sections was assumed to be fixed in the RP, this did not and could not lead to any errors (the "second" optimum solution discovered in [9], as we have mentioned above, is the result of an ambiguity). Of course, this does not rule out technological solutions where the lubricant is supplied (for instance, in order to maintain the pressure level, preventing cavitation) along "supply channels" in the bearing base in certain sections of the gap. In that case, however, it is better not to fix the pressure *a priori*, but rather take it from the solution of the corresponding variational problem.

The optimized parameter is not improved by the addition of further constraints which reduce the number of possible solutions, and in fact as a rule is made worse than in the original problem. This will happen, for instance, if a "linear" mass of lubricant—$m$ (per unit length of shaft) is specified in the gap. Since this possibility is considered in direct calculations of the flow in closed journal bearings [1],

Table 1

| $q$ | $x_a \times 10^3$ | $x_b \times 10^3$ | $x_c \times 10^3$ | $x_f \times 10^3$ | $h_a = h_f$ | $h_{\min}$ | $C_N \times 10^2$ | $C_M \times 10$ |
|---|---|---|---|---|---|---|---|---|
| 1.21 | 586 | 586 | 914 | 914 | 1.81 | 1.00 | 1.345 | 1.441 |
| 1.30 | 592 | 648 | 852 | 908 | 1.94 | 1.00 | 1.282 | 1.359 |
| 1.34 | 593 | 670 | 830 | 907 | 2.00 | 1.00 | 1.230 | 1.323 |
| 1.35 | 593 | 676 | 824 | 907 | 2.03 | 1.00 | 1.211 | 1.310 |
| 1.40 | 594 | 697 | 803 | 906 | 2.10 | 1.00 | 1.144 | 1.269 |
| 1.50 | 594 | 742 | 758 | 906 | 2.25 | 1.00 | 1.006 | 1.189 |
| 1.60 | 594 | – | – | 906 | 2.40 | 1.06 | 0.880 | 1.110 |
| 2.00 | 594 | – | – | 906 | 3.00 | 1.33 | 0.566 | 0.892 |
| 1.43 | 594 | 719 | 781 | 906 | 2.00 | 1.00 | 1.077 | 1.241 |
| 1.47 | 595 | – | – | 905 | 2.00 | 1.01 | 1.000 | 1.203 |
| 1.5 | 595 | – | – | 905 | 2.00 | 1.03 | 0.940 | 1.176 |
| 1.6 | 598 | – | – | 902 | 2.00 | 1.14 | 0.750 | 1.090 |
| 1.8 | 610 | – | – | 890 | 2.00 | 1.40 | 0.380 | 0.943 |
| 2.0 | – | – | – | – | 2.00 | 2.00 | 0 | 0.830 |

the solution of variational problems with fixed $m$ is of interest. Let $\nu$ be a constant Lagrange multiplier, which introduces an integral representation for $m$ in the Lagrange functional $J$, and $m_0$ is the value of $m$ obtained in the original variational problem. Then, omitting the details, it can be shown that the best value of the optimized functional corresponding to $m = m_0$ is obtained for $\nu = 0$. We note, finally, that solutions obtained for a journal bearing of infinite length in the direction of its axis are not only of theoretical interest. Their use is quite justified in journal bearings of finite length in which it is difficult (in the ideal case, impossible) for the lubricant to spread in axial direction. This can be achieved by the use of lateral washers and seals [5].

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